Quantum field theory methods in non-linear stochastic dynamics

Butuza Andrei Alexandru, Romania
Babes-Bolyai University of Cluj Napoca
Supervised by: Prof. M. Hnatic and Dr. R. Remetsky

Bogoliubov Laboratory of Theoretical Physics

July 21, 2017

Overview

Introduction-Main problem

- Turbulent flow $\rightarrow$ mathematical difficulty
- Fully developed turbulent flow $\rightarrow$ R_e tends toward infinity
- Use of Quantum field theory (QFT) and Renormalization Group (RG)
- Prandtl number is a parameter of turbulent flow.

Incompressible Navier-Stokes and Advection diffusion

$$\partial_t \ \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu_0 \Delta \mathbf{v} + \nabla P = f^v$$

$$\partial_t \phi + (\phi \nabla)\phi - \nu_0 \mu_0 \Delta \phi = 0$$

- $\mathbf{v}$ is the fluctuating part of the velocity field
- ν_0 is kinematic viscosity
- P is pressure
- f^v is external random force
- μ_0 is the inverse Prandtl number
- Incompressible $\rightarrow$ $\nabla \cdot \mathbf{v} = 0$

Incompressible Navier-Stokes and Advection diffusion

The correlation function maintains the steady state while pumping energy in the system from large scale:

$$\langle f_i(x) f_j(x') \rangle = \delta(t-t')(2\pi)^{-d} \int d\mathbf{k} P_{ij}(\mathbf{k}) e^{-i\mathbf{k}(\mathbf{x}\text{-}\mathbf{x}')} d_f(k)$$

where $d_f(k) = g_0 \nu_0^3 k^{4-d-2\epsilon}$, ϵ being an arbitrary parameter and

$$P_{ij}(\mathbf{k}) = \delta_{i,j} - \frac{k_i k_j}{k^2}$$

is the transverse projector.

Physical value $\rightarrow$ $\epsilon = 2$

Prandtl number

Molecular Prandtl number

Prandtl number is a dimensionless parameter defined as the ratio of molecular viscosity to molecular diffusivity

$$P_r = \frac{\nu}{\alpha} = \frac{1}{\mu_*}$$

Turbulent Prandtl number

Also a dimensionless parameter, Turbulent Prandtl number is defined as the ratio of turbulent viscosity to turbulent diffusivity.

In a fully developed turbulent system, Turbulent Prandtl number is an universal constant

QFT approach

We can write the action functional S from the stochastic equations:

$$S = \frac{1}{2}\mathbf{v}'\Delta\mathbf{v}' + \mathbf{v}[-\partial_t\mathbf{v} + \nu_0\Delta\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}] + \phi'[-\partial_t\phi + \mu_0\nu_0\Delta\phi - (\mathbf{v}\cdot\nabla)\phi]$$

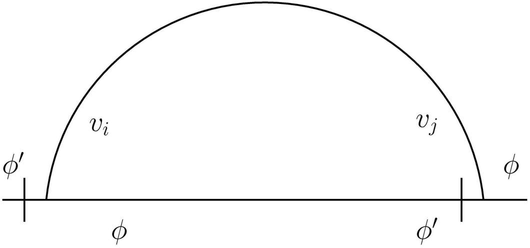
From which the propagators are extracted:
$\Delta_{ij}^{vv}(k) = \frac{g_0\nu_0^3 k^{4-d-2\epsilon}}{(i\omega_k+\nu_0 k^2)(-i\omega_k+\nu 0k^2)}$; $\Delta_{ij}^{v'v}(k) = \frac{P_{ij}(k)}{i\omega_k+\nu_0 k^2}$; $\Delta^{\phi'\phi}(k) = \frac{1}{i\omega_k+\nu_0\mu_0 k^2}$

The vertices are:

$$V_j^{\phi}(k) = ik_j$$

$$V_{ijs}^{v}(k) = i(k_j\delta_{is} + k_s\delta_{ij})$$

QFT approach-Scalar field

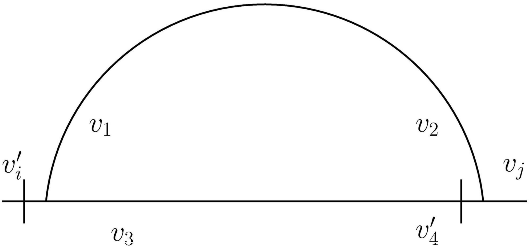


$$\langle \phi' \phi \rangle = \int \int d^d k d\omega_k V_1^{\phi}(p) V_2^{\phi}(p-k) \Delta_{ij}^{vv}(k) \Delta^{\phi\phi'}(p-k)$$

The integral can be solved analitically using residue theorem, Taylor expanions and approximations and a change of coordinates and yields:

$$\langle \phi' \phi \rangle = -\frac{S_d g_0 \nu_0 (d-1)}{(2\pi)^d 4\epsilon d (1+\mu_0)} m^{-2\epsilon} p^2$$

QFT approach-Vector field



$$\langle v'_i v_j \rangle = \int\int d^d k d\omega_k V^v_{i13}(p) V^v_{4j2}(p-k) \Delta^{vv}_{12}(k) \Delta^{vv'}_{34}(p-k)$$

The integral can be solved analitically using residue theorem, Taylor expansions and approximations and a change of coordinates and yields:

$$\langle v'_i v_j \rangle = -\frac{S_d g_0 \nu_0 (d-1)}{(2\pi)^d 8\epsilon (d+2)} m^{-2\epsilon} p^2 \delta_{i,j}$$

Renormalization

Using RG we have a renormalized action functional S_R:

$$S_R = \frac{1}{2}\mathbf{v}'\Delta\mathbf{v}' + \mathbf{v}[-\partial_t\mathbf{v} + \nu Z_1\Delta\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}] + \phi'[-\partial_t\phi + \mu\nu Z_2\Delta\phi - (\mathbf{v}\cdot\nabla)\phi]$$

Where Z_1 and Z_2 are obtained from: $\nu_0 = \nu Z_\nu$; $\mu_0 = \mu Z_\mu$; $g_0 = g\mu^{2\epsilon}Z_g$

and $Z_\nu = Z_1 \ Z_g = Z_1^{-3} \ Z_\mu = Z_2 Z_1^{-1}$

By using some beta functions $\beta_g and \beta_\mu$ and imposing $\beta_{g^*} = \beta_{\mu^*} = 0$ we are looking for infrared fixed points g^* and μ^*

Results

$$P_{rt} = \frac{1}{\mu^*}$$

By solving the quadratic equation for μ^* and taking the positive value:

$$\mu^*(1 + \mu^*) = \frac{2(d+2)}{d}$$

we get $\mu^* \approx 1.393$ for $d = 3$ which fits in the experimental range for Prandtl number $\langle 0.7\ 0.9 \rangle$